Completeness in Natural Logic: What and Why?

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March 20, 2005

1 Introduction

My interest in the topic of proof theory and semantics comes from several recent experiences teaching the most basic ideas of semantics to graduate students in cognitive science, as part of a broader course. Certainly model theoretic semantics leads to an interesting set of questions, and that is for the good. But the topic of this meeting is a certain limitation on the overall enterprise, and so this is what I want to emphasize here. One motivates the subject as an account of the intuitive concept of *entailment*. But then after providing semantics for various fragments, one rarely goes back and does the hard work of determining the complete logics for validity in those fragments. Perhaps textbook accounts in linguistics fail to do this because it is hard and/or requires specialized methods from logic.[1] In any case, the first point to make is that a model-theoretic semantics could, and should, lead to complete proof systems for fragments of natural language. The second point then follows: if one is seriously interested in entailment, why not study it axiomatically instead of building models? In particular, if one has a complete proof system, why not declare it to *be* the semantics? Indeed, why should semantics be founded on model theory rather than proof theory? This last perspective should lead to several lines of research for natural language semantics in settings where model theory has been difficult. I think about the semantics of generics and perhaps also about intensional phenomena. Once researchers feel free to propose semantics in proof-theoretic style, it should be possible to think again about these topics, and perhaps make progress or at least ask new questions.

2 Some Complete Systems for Simple Fragments

It is possible to get complete systems for simple fragments of natural language. I am in fact interested in getting complete systems for as large a fragment as possible. Here is what I know about this now. Consider sentences S of the following forms: *All* X *are* Y (with X and Y common nouns, say in the plural, such as *students*, *plumbers*, etc.), *Some* X *are* Y, *No* X *are* Y, J *is an* X (with J a name like *John*), and J *is* M. Then the proof rules in Figure 1 are complete. This system is a Hilbert-style system, but shortly we'll see how ideas from natural deduction lead to an interesting observation. Moreover, one can add boolean

[1] This is not the case for papers in some neighboring areas. I intend to summarize the complete proof systems in the 10-20 papers in the natural language processing area that seem relevant to the topic, and the fewer number of papers on modern reconstructions of syllogistic reasoning. None of them seems especially proof-theoretic, however.

connectives between sentences to the fragment, and then in the proof system one adds some standard axiom schemes for those connectives. (This would give negative sentences, and so the results here subsume *syllogistic reasoning*.

All of this work is fairly technical, and I couldn't summarize the completeness arguments succinctly. One has to work with fragments of this fragment. In the simplest cases the arguments are new and ad hoc, and using them one builds up to the whole fragment where the arguments resemble more standard arguments from logic.

To make this brief, I didn't say what the semantics is that makes for a complete system. It is the natural one. I would think that any reasonable semantics could be used. Indeed, this echoes the point above that we could (and perhaps should) take the semantics of the fragment to be the proof theory, since it gives a full account of *inference*.

The natural theoretical question here is whether a complete proof system *induces* the model theoretic semantics in any sense. That is, suppose we don't know the meaning of *All*, *Some*, etc., but we do know that the proof system is the full account of inference. Can we then *solve for* the semantics? Would additional assumptions be needed?

Going beyond this one would like to add more interesting words. I have investigated the situation in several directions. One worth mentioning is the non-first-order quantifier *most*. It seems that nobody ever worked out the complete proof system for *most* on top of the very simple fragment that we have so far. I am nearly done with a completeness proof for the system of Figures 1 and 2. Here the semantics is that *Most X are Y* means that the number of things in both X and Y is strictly larger than the number in X but not Y. (We only use finite models.) The completeness work seems difficult because the whole thing is very "non-canonical". To get a feeling for this, try to decide whether

$$\textit{All } U \textit{ are } V, \textit{Most } U \textit{ are } V, \textit{All } W \textit{ are } V, \textit{Most } V \textit{ are } W, \textit{Most } W \textit{ are } U \models \textit{Most } U \textit{ are } W.$$

or not. And as you can see from the system, most of the axioms are not properties that anyone would immediately think of as properties of *most*. So here we have a sharper question: suppose the proof-theoretic account of meaning turned out very different than the model-theoretic account. How would we judge between the two?

Natural Deduction For purposes of the proof-theoretic view, it is nice to know that the basic fragment of Figure 1 may be reformulated as a natural deduction system. Briefly, we have interesting introduction rules which ask us to withdraw premises. For example, our system has the following deduction which shows the value of *names* in reasoning:

$$\dfrac{\dfrac{\textit{All } Y \textit{ are } Z \qquad \dfrac{\textit{All } X \textit{ are } Y \quad [J \textit{ is an } X]_*}{J \textit{ is a } Y}}{J \textit{ is a } Z}}{\textit{All } X \textit{ are } Z}\;*$$

The step marked $*$ involves withdrawal of a premise. So in this system we deduce the classical syllogism Barbara rather than take it as an axiom. We also would like

$$\dfrac{\textit{Some } X \textit{ are } Y \qquad \begin{array}{c}[\textit{All } U \textit{ are } V]\\ \vdots \\ \textit{No } X \textit{ are } Y\end{array}}{\textit{Some } U \textit{ are } V}$$

$$\frac{}{All\ X\ are\ X} \qquad \frac{All\ X\ are\ Z \quad All\ Z\ are\ Y}{All\ X\ are\ Y}$$

$$\frac{Some\ X\ are\ Y}{Some\ Y\ are\ X} \qquad \frac{Some\ X\ are\ Y}{Some\ X\ are\ X}$$

$$\frac{All\ Y\ are\ Z \quad Some\ X\ are\ Y}{Some\ X\ are\ Z} \qquad \frac{}{J\ is\ J}$$

$$\frac{M\ is\ J}{J\ is\ M} \qquad \frac{J\ is\ M \quad M\ is\ F}{J\ is\ F}$$

$$\frac{J\ is\ an\ X \quad J\ is\ a\ Y}{Some\ X\ are\ Y} \qquad \frac{All\ X\ are\ Y \quad J\ is\ an\ X}{J\ is\ a\ Y}$$

$$\frac{M\ is\ an\ X \quad J\ is\ M}{J\ is\ an\ X} \qquad \frac{All\ X\ are\ Z \quad No\ Z\ are\ Y}{No\ X\ are\ Y}$$

$$\frac{No\ X\ are\ Y}{No\ Y\ are\ X} \qquad \frac{No\ X\ are\ X}{No\ X\ are\ Y}$$

$$\frac{No\ X\ are\ X}{All\ X\ are\ Y} \qquad \frac{Some\ X\ are\ Y \quad No\ X\ are\ Y}{S}$$

Figure 1: The rules of the system in the fragment mentioned in Section 1.

$$\frac{\textit{Most } X \textit{ are } Y \quad \textit{All } Y \textit{ are } Z}{\textit{Most } X \textit{ are } Z} \qquad \frac{\textit{Most } Y \textit{ are } Z \quad \textit{All } X \textit{ are } Y \quad \textit{All } Y \textit{ are } X}{\textit{Most } X \textit{ are } Z}$$

$$\frac{\textit{All } Z \textit{ are } Y \quad \textit{All } Y \textit{ are } X \quad \textit{Most } X \textit{ are } Z}{\textit{Most } Y \textit{ are } Z}$$

$$\frac{\textit{All } U \textit{ are } X \quad \textit{Most } X \textit{ are } V \quad \textit{All } V \textit{ are } Y \quad \textit{Most } Y \textit{ are } U}{\textit{Some } U \textit{ are } V}$$

$$\frac{\textit{Most } X \textit{ are } Y}{\textit{Some } X \textit{ are } Y} \qquad \frac{\textit{All } X \textit{ are } Y \quad \textit{Some } X \textit{ are } X}{\textit{Most } X \textit{ are } Y} \qquad \frac{\textit{Most } X \textit{ are } Y \quad \textit{Most } X \textit{ are } Z}{\textit{Some } Y \textit{ are } Z}$$

Figure 2: The logic of *Most* X *are* Y.

This system better reflects actual reasoning with hypothetical premises and reductio ad absurdum arguments.

3 Less Linguistically Trivial Fragments

Obviously the fragments mentioned so far are I am currently working on so simple that one wonders whether the proof systems will "scale up". I think it should be possible to extend the work to fragments will real verbs instead of just the copula. I have systems that do this which are defined in terms of a simple categorial grammar encorporating relative clauses allowing peripheral extractions only, so that the relative clauses themselves are of categories S/NP or NP\S. One also can add reflexive pronouns and other grammatical devices like possessives that have a smooth treatment in the fragment. For example, one of the rules in the system would be

$$\frac{\textit{Every } X \textit{ who } A \ B \quad J \textit{ is an } X \quad J \ A}{J \ B}$$

where A and B are of category NP\S.

Once again, the first goal is simply to give an account of validity in the fragment. In this particular fragment, because we have quantifier scope ambiguities, we are most naturally lead (I think) to replace sentences by their parse trees (or some equivalent notion).

Incidentally, another option for this work is to translate the fragment into the typed lambda calculus (following Montague), and then to ask what the complete logic is for the fragment *interpreted on the class of models that satisfy the natural meaning postulates.* Here I don't believe this work has ever been done, and it should not be so hard to do it.

4 Beyond Traditional Proof Theory

My feeling is that it should be possible to design complete proof theories for the many of the extensional phenomena that one usually studies in semantics. However, there will be quite a few new problems that arise, mainly because the whole enterprise of proof theory was never intended to for this purpose, and so most of the work that has been done in it is not going to be applicable. For example, when one considers anaphora, say as treated in "dynamic" frameworks like DPL, the question immediately comes up as to how to get insightful proof-theoretic treatments. One would have to re-think the overall proof systems.

Things get even more interesting (and speculative) when one tries to use the approach to really supplement or even supplant model-theoretic semantics. One can envision using logics presented proof theoretically without a semantics, as one finds in treatments of *Typically X are Y* in the AI literature on defeasible reasoning. So in this approach, one takes the proof system as primary, and afterwards asks for the semantics (if there is one). My hunch is that there will be linguistic problems connected with matters like vagueness and generics, problems much more difficult and serious than the ones considered in defeasible reasoning. And so the whole matter of *proof-theoretic semantics* will face challenging problems in this area.